PDL Model Checking of Parse Forests

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Abstract

Due to its kinship with XPath, satisfiability issues for propositional dynamic logic (PDL) on trees in presence of some tree language have been extensively studied in the XML community. We describe two applications of the same problem for model-theoretic syntax and compilers construction, where the tree language is the set of parse trees of some word according to a context-free grammar. This new variant is still complete for exponential time, but we explore the impact of natural grammar restrictions and establish complexities ranging from nondeterministic polynomial time to polynomial space in the relevant cases.

1 Introduction

Although originally motivated by applications in computational linguistics (Kracht, 1995; Palm, 1999; Afanasiev et al., 2005), *propositional dynamic logic on trees* ($\mathrm{PDL}_{\mathsf{tree}}$) has been extensively studied in the XML community (Marx, 2005; Benedikt et al., 2008; ten Cate and Segoufin, 2010), where it is better known as *Regular XPath*. A prominent algorithmic problem in this context is the satisfiability of formulæ in presence of a tree language, the latter being typically described by a DTD: given a formula φ and a tree language L, does there exist a tree t in L which is also a model of φ? Benedikt et al. (2008) comprehensively investigate this topic, and in some restricted cases the problem becomes tractable (Montazerian et al., 2007; Ishihara et al., 2009).

In this paper, we investigate a variant of the problem, where the tree language L we consider is a *parse forest*, i.e. the set of parse trees of a word w according to a context-free grammar G. More precisely, after recalling the formal apparatus of $\mathrm{PDL}_{\mathsf{tree}}$ in Section 2, we present in Section 3 two applications of the parse forest model-checking problem (PFMC):

- in computational linguistics, where we advocate a mixed approach for model-theoretic syntax (Pullum and Scholz, 2001), with syntactic structures described by the conjunction of a grammar with a $\mathrm{PDL}_{\mathsf{tree}}$ constraint, and
- in compilers construction, where $\mathrm{PDL}_{\mathsf{tree}}$ formulæ provide an compelling means for parser disambiguation (Thorup, 1994; Klint and Visser, 1994; Kats et al., 2010).

These applications motivate (1) practically relevant restrictions on the grammar G, which have no natural counterpoint in the XML literature, and (2) considering the full logic $\mathrm{PDL}_{\mathsf{tree}}$ rather than the weaker $\mathrm{PDL}_{\mathsf{core}}$ fragment (aka Core XPath) employed in XML processing.

We map the resulting complexity landscape for the problem in Section 4. Although the general case is ExpTime-complete like the classical problem, our cases of interest have more affordable PSpace-complete and NPTime-complete complexities (see Figure 3 for a summary). A somewhat surprising corollary for model-theoretic syntax is that the *recognition problem* for $\mathrm{PDL}_{\mathsf{tree}}$, i.e. whether there exists a tree model with the input word as yield, is PSpace-complete if empty labels are forbidden—the best algorithms for this were only known to operate in exponential time (Cornell, 2000; Palm, 2004).

2 Propositional Dynamic Logic on Trees

Propositional dynamic logic (PDL, see (Fischer and Ladner, 1979)) is a modal logic where "programs"—in the form of regular expressions over the relations in a frame—are used as modal operators. In the case of ordered trees (Kracht, 1995; Afanasiev et al., 2005; ten Cate and Segoufin, 2010), the two relations are the *child* relation $\downarrow$ between a parent node and any of its immediate children, and the *right-sibling* relation $\rightarrow$ between a node and its immediate right sibling.

2.1 Syntax and Semantics

Formally, a $\mathrm{PDL}_{\mathsf{tree}}$ formula φ is defined by the abstract syntax

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\pi\rangle\varphi \qquad \text{(node formulæ)}$$

$$\pi ::= \downarrow \mid \rightarrow \mid \pi\,;\pi \mid \pi + \pi \mid \pi^* \mid \pi^{-1} \mid \varphi? \qquad \text{(path formulæ)}$$

where p is an atomic proposition ranging over some countable set AP—because we only deal with satisfiability questions, we can actually assume AP to be finite. We enrich this syntax as usual by defining box modalities as duals $[\pi]\varphi \stackrel{\text{def}}{=} \neg\langle\pi\rangle\varphi$ of the diamond ones, inverses to the atomic path formulæ as $\uparrow \stackrel{\text{def}}{=} \downarrow^{-1}$ and $\leftarrow \stackrel{\text{def}}{=} \rightarrow^{-1}$, and boolean connectives $\bot \stackrel{\text{def}}{=} \neg\top$, $\varphi_1 \vee \varphi_2 \stackrel{\text{def}}{=} \neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \supset \varphi_2 \stackrel{\text{def}}{=} \neg\varphi_1 \vee \varphi_2$, and $\varphi_1 \equiv \varphi_2 \stackrel{\text{def}}{=} (\varphi_1 \supset \varphi_2) \wedge (\varphi_2 \supset \varphi_1)$.

Formulæ are interpreted over *finite ordered trees* t with nodes labeled by subsets of AP. Such a tree t is a partial function from the set $\mathbb{N}^*$ of finite sequences of natural numbers to AP, s.t. its *domain* $\mathsf{dom}\,t$ is (1) *finite*, (2) *prefix closed*, i.e. uv in $\mathsf{dom}\,t$ for some u, v in $\mathbb{N}^*$ implies that u is also in $\mathsf{dom}\,t$, and (3) *predecessor closed*, i.e. if ui is in $\mathsf{dom}\,t$ for some u in $\mathbb{N}^*$ and i in $\mathbb{N}$, then uj is also in $\mathsf{dom}\,t$ for all $j < i$ in $\mathbb{N}$. Such a tree can be seen as a structure $\mathfrak{M}_t = \langle \mathsf{dom}\,t, \downarrow_t, \rightarrow_t, t\rangle$ with

$$\downarrow_t \stackrel{\text{def}}{=} \{(u, ui) \mid ui \in \mathsf{dom}\,t\} \qquad \rightarrow_t \stackrel{\text{def}}{=} \{(ui, u(i+1)) \mid u(i+1) \in \mathsf{dom}\,t\}\,.$$

We define the *interpretations* of $\mathrm{PDL}_{\mathsf{tree}}$ formulæ over t inductively by

$$\begin{array}{lll}
[\![\top]\!]_t \stackrel{\text{def}}{=} \mathsf{dom}\,t & [\![\neg\varphi]\!]_t \stackrel{\text{def}}{=} \mathsf{dom}\,t \setminus [\![\varphi]\!]_t & [\![p]\!]_t \stackrel{\text{def}}{=} \{u \in \mathsf{dom}\,t \mid p = t(u)\} \\
[\![\varphi_1 \wedge \varphi_2]\!]_t \stackrel{\text{def}}{=} [\![\varphi_1]\!]_t \cap [\![\varphi_2]\!]_t & [\![\langle\pi\rangle\varphi]\!]_t \stackrel{\text{def}}{=} [\![\pi]\!]_t^{-1}([\![\varphi]\!]_t) & [\![\varphi?]\!]_t \stackrel{\text{def}}{=} \{(u,u) \mid u \in [\![\varphi]\!]_t\} \\
[\![\downarrow]\!]_t \stackrel{\text{def}}{=} \downarrow_t & [\![\rightarrow]\!]_t \stackrel{\text{def}}{=} \rightarrow_t & [\![\pi_1\,;\pi_2]\!]_t \stackrel{\text{def}}{=} [\![\pi_1]\!]_t \fatsemi [\![\pi_2]\!]_t \\
[\![\pi^*]\!]_t \stackrel{\text{def}}{=} [\![\pi]\!]_t^* & [\![\pi^{-1}]\!]_t \stackrel{\text{def}}{=} [\![\pi]\!]_t^{-1} & [\![\pi_1 + \pi_2]\!]_t \stackrel{\text{def}}{=} [\![\pi_1]\!]_t \cup [\![\pi_2]\!]_t
\end{array}$$

Observe that these are sets of nodes included in $\mathsf{dom}\, t$ in the case of node formulæ, but binary relations included in $\mathsf{dom}\, t \times \mathsf{dom}\, t$ in the case of path formulæ; thus $[\![\pi]\!]_t^*$ denotes the reflexive transitive closure of $[\![\pi]\!]_t$ and $[\![\pi]\!]_t^{-1}$ its inverse, while $[\![\pi_1]\!]_t \fatsemi [\![\pi_2]\!]_t$ denotes the composition of the two relations $[\![\pi_1]\!]_t$ and $[\![\pi_2]\!]_t$. A node u in $\mathsf{dom}\, t$ *satisfies* φ, noted $t, u \models \varphi$, if u is in $[\![\varphi]\!]_t$. A tree t satisfies φ, noted $t \models \varphi$, if its root ε satisfies φ; we let $[\![\varphi]\!] \stackrel{\text{def}}{=} \{t \mid t \models \varphi\}$ be the set of models of φ.

Example 1 (Basic Navigation)**.** Several simple formulæ helping navigation can be defined: $\mathsf{root} \stackrel{\text{def}}{=} \neg\langle\uparrow\rangle\top$ holds only at the root, $\mathsf{leaf} \stackrel{\text{def}}{=} \neg\langle\downarrow\rangle\top$ only at a leaf node, $\mathsf{first} \stackrel{\text{def}}{=} \neg\langle\leftarrow\rangle\top$ at a leftmost one, and $\mathsf{last} \stackrel{\text{def}}{=} \langle\rightarrow\rangle\top$ at a rightmost one.

We can also define the *first-child* relation $\swarrow \stackrel{\text{def}}{=} \downarrow;\mathsf{first}?$, and conversely express the child relation as $\downarrow \equiv \swarrow;\rightarrow^*$: this shows that we could work on binary tree models instead of the unranked ones we used in our definitions.

Example 2 (Parse Trees (Blackburn et al., 1993))**.** Recall that a *context-free grammar* (CFG) is a tuple $G = \langle N, \Sigma, P, S\rangle$ composed of a finite *nonterminal* alphabet N, a finite *terminal* alphabet Σ disjoint from N and forming a *vocabulary* $V \stackrel{\text{def}}{=} N \uplus \Sigma$, a finite set of *productions* $P \subseteq N \times V^*$, and an *axiom* $S \in N$. We denote the empty sequence by ε and write $\Sigma' \stackrel{\text{def}}{=} \Sigma \uplus \{\varepsilon\}$ and $V' \stackrel{\text{def}}{=} V \uplus \{\varepsilon\}$.

Given a context-free grammar G, its set of parse trees forms a local tree language, which can be expressed as $[\![\varphi_G]\!]$ for a $\mathrm{PDL}_{\mathsf{tree}}$ formula φ_G with V' as set of atomic propositions. First define a path formula π_α that defines a sequence of sibling nodes labeled by α in V^*:

$$\pi_\alpha \stackrel{\text{def}}{=} \begin{cases} X?;\rightarrow;\pi_{\alpha'} & \text{if } \alpha = X\alpha', X \in V, \alpha' \neq \varepsilon\,, \\ X?;\mathsf{last}? & \text{if } \alpha = X \in V\,, \\ \varepsilon?;\mathsf{last}? & \text{otherwise, i.e. if } \alpha = \varepsilon\,. \end{cases}$$

$$\begin{aligned} \varphi_G \stackrel{\text{def}}{=}\ & S && \text{(the root is labeled by } S\text{)} \\ & \wedge [\downarrow^*]\big(\mathsf{leaf} \equiv \bigvee_{a \in \Sigma'} a && \text{(leaves are terminals and internal nodes nonterminals)} \\ & \qquad \wedge \bigwedge_{A \in V} A \supset \bigvee_{A \to \alpha} \langle\swarrow;\pi_\alpha\rangle\top\big)\,. && \text{(productions are enforced)} \end{aligned}$$

2.2 The Conditional Fragment

We will consider in this paper several fragments of $\mathrm{PDL}_{\mathsf{tree}}$, most importantly the *conditional path* fragment $\mathrm{PDL}_{\mathsf{cp}}$ (Palm, 1999; Marx, 2005), with a restricted syntax on path formulæ

$$\begin{aligned} \pi &::= \alpha \mid \pi\,;\pi \mid \pi + \pi \mid \varphi? \mid (\alpha;\varphi?)^* && \text{(conditional paths)} \\ \alpha &::= \leftarrow \mid \rightarrow \mid \uparrow \mid \downarrow\,. && \text{(atomic paths)} \end{aligned}$$

This fragment is of particular relevance, because it extends the *core language* $\mathrm{PDL}_{\mathsf{core}}$ (Blackburn et al., 1996; Gottlob and Koch, 2002) (which features α^* instead of $(\alpha;\varphi?)^*$) and captures exactly first-order logic over finite ordered trees with the two relations $\rightarrow^+$ and $\downarrow^+$ (Marx, 2005).

Example 3 (General Successor)**.** Observe that the formulæ in examples 1 and 2 are actually in $\mathrm{PDL}_{\mathsf{core}}$. The *general successor* relation $\prec \stackrel{\text{def}}{=} (\mathsf{last}?;\uparrow)^*;\rightarrow;(\downarrow;\mathsf{first}?)^*$ is an example of a path that is not definable in $\mathrm{PDL}_{\mathsf{core}}$—this can be checked for instance using an Ehrenfeucht Fraïssé argument.

We denote by $\mathrm{PDL}_{\mathsf{tree}}[\downarrow]$ (resp. $\mathrm{PDL}_{\mathsf{cp}}[\downarrow]$, $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$) the fragments with only downward navigation, i.e. without the $\rightarrow$, $\leftarrow$, and $\uparrow$ atomic paths.

3 Model-Checking Parse Forests

Many problems arising naturally with $\mathrm{PDL}_{\mathsf{tree}}$ are decidable, notably the

model-checking problem: given a tree t and a formula φ, does $t \models \varphi$? This is known to be in PTime even for larger fragments of PDL (Lange, 2006).

satisfiability problem: given a formula φ, does there exist a tree t s.t. $t \models \varphi$? This is known to be ExpTime-complete (Afanasiev et al., 2005).

In the context of XML processing and XPath, an intermediate question between model-checking and satisfiability also arises:

satisfiability in presence of a tree language: given a formula φ and a regular tree language L, does there exist a tree $t \in L$ s.t. $t \models \varphi$?

Due to its initial XML motivation, the basic case for this problem is that of a $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ formula (a downward Core XPath query) and of a local tree language (described by a DTD), but many variants exist (Benedikt et al., 2008; Montazerian et al., 2007; Ishihara et al., 2009). Our own flavour is motivated by applications in computational linguistics and programming languages, where the tree language is the set of parse trees of a word w in Σ^* according to a CFG $G = \langle N, \Sigma, P, S\rangle$ verifying $V' \stackrel{\text{def}}{=} \Sigma \uplus N \uplus \{\varepsilon\} = \mathsf{AP}$.

More precisely, following a well-known construction of Bar-Hillel et al. (1961), if $w = a_1 \cdots a_n$ is a word of length n, the set of parse trees or *parse forest* of a CFG G for w, written $L_{G,w}$, is the regular tree language recognized by a tree automaton $\mathcal{A}_{G,w}$ with state set

$$Q_{G,w} \stackrel{\text{def}}{=} \{(i, X, j) \mid 0 \leq i \leq j \leq n, X \in V'\} ,$$

alphabet V', initial state $(0, S, n)$, and rules

$$\delta_{G,w} \stackrel{\text{def}}{=} \{(i_0, A, i_m) \to A((i_0, X_1, i_1) \cdots (i_{m-1}, X_m, i_m)) \mid A \to X_1 \cdots X_m \in P \wedge 0 \leq i_0 \leq \cdots \leq i_m \leq n\}$$
$$\cup \{(i, a_{i+1}, i+1) \to a_{i+1}() \mid 0 \leq i < n\} \cup \{(i, \varepsilon, i) \to \varepsilon() \mid 0 \leq i \leq n\} .$$

Intuitively, a state (i, X, j) of this automaton recognizes the set of trees derivable in G from the symbol X and spanning the factor $a_{i+1} \cdots a_j$ of w.

Parse Forest Model-Checking Problem (PFMC)**.**

input a context free grammar G, a word w, and a $\mathrm{PDL}_{\mathsf{tree}}$ formula φ,

question does there exists $t \in L_{G,w}$ s.t. $t \models \varphi$? □

Note that the automaton $\mathcal{A}_{G,w}$ has size $O(|G| \cdot |w|^{m+1})$ if m is the maximal length of a production rightpart in G; since the grammar can be put in quadratic form (corresponding to the binarization we would also perform on the formula), this typically results in size $O(|G| \cdot |w|^3)$. Therefore, although a tree automaton for the tree language is not part of the input, it can nevertheless be constructed

in logarithmic space. The originality of the problem stems from considering parse forests, which form a rather restricted class of tree languages.

In Section 4, we will investigate the complexity of this problem, and focus on the influence of the acyclicity and ε-freeness of G: Define the *derivation* relation $\Rightarrow$ between sequences in V^* by $\beta A\gamma \Rightarrow \beta\alpha\gamma$ iff $A \to \alpha$ is a production of G and β, γ are arbitrary sequences in V^*. A CFG is *acyclic*, if none of its nonterminals A allows $A \Rightarrow^+ A$. A CFG is *ε-free*, if none of its productions is of form $A \to \varepsilon$ for some nonterminal A.

In the remainder of this section, we motivate the problem by considering applications in computational linguistics (Section 3.1) and compilers construction (Section 3.2).

3.1 Application: Computational Linguistics

In contrast with many formal theories of syntax that describe natural language sentences through "generative-enumerative means", Pullum and Scholz (2001) champion *model-theoretic syntax*, where the syntactic structures (typically, trees) of a natural language are the models of some logical formula. They point out interesting consequences on theories of syntax, but here we thoroughly betray the spirit of their work in exchange for some practicality.

Indeed, the usual approach to model-theoretic syntax would be to describe a language through a *huge* formula φ of $\mathrm{PDL}_{\mathsf{tree}}$ or monadic second-order logic (MSO) on trees. Checking whether a given sentence w can be assigned a structure then reduces to a recognition problem on a tree automaton $\mathcal{A}_\varphi$ of exponential (for $\mathrm{PDL}_{\mathsf{tree}}$) or non-elementary (for MSO) size (Cornell, 2000).

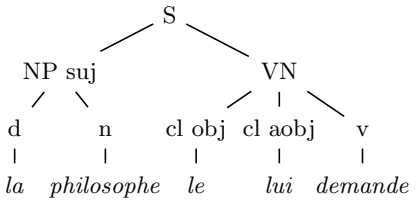
A Mixed Approach. We consider a pragmatic approach, where

- a CFG describes the *local* aspects of syntax, e.g. that a canonical transitive French sentence can be decomposed into a noun phrase acting as subject followed by a verb kernel and an object noun phrase corresponds to a production $\mathrm{S} \to \mathrm{NP\,VN\,NP}$, while
- *long-distance* dependencies and more complex linguistic constraints are described through $\mathrm{PDL}_{\mathsf{tree}}$ formulæ.

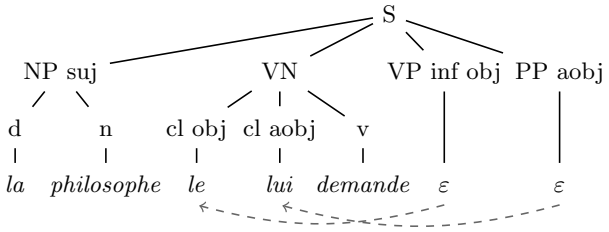
Example 4 (French Clitics)**.** A toy grammar for French sentences with predicative verbs like "dire" or "demander" could look like (in an extended syntax where $X?$ describes zero or one occurrences of symbol X):

S → NPsuj? VN VPinfobj? PPaobj?	d → *la*
NPsuj → d n	n → *philosophe*
VN → clsuj? clobj? claobj? v	v → *demande* | *réfléchir*
VPinfobj → *de* VN	clsuj → *elle*
PPaobj → *à* NP	clobj → *le*
	claobj → *lui*

Such predicative verbs have a mandatory object and subject, and an optional indirect object. But all three canonical arguments can be replaced by *clitics* in the verb matrix VN. This grammar fragment generates reasonable sentences like



(a) Syntax tree according to Example 4.



(b) Analysis with moved constituents.

Figure 1: Syntax trees for "la philosophe le lui demande."

La philosophe demande de réfléchir. (The philosopher asks to think.)
La philosophe le lui demande. (The philosopher asks it to her.)

where the "le" clitic acts as direct object and "lui" as an indirect one (see Figure 1a for an example syntax tree). It also generates ungrammatical ones like

* Elle le lui demande de réfléchir. (She asks it to her to think.)
* demande. (asks.)

where there are duplicated or missing arguments.

Instead of refining the grammar (which might prove impossible, for instance if it was automatically extracted from a treebank, i.e. a set of sentences annotated with syntactic trees), we can filter out the unwanted trees using a $\mathrm{PDL}_{\mathsf{tree}}$ formula. To improve readability, we take symbols like "VPinfobj" or "clsuj" to denote sets of atomic propositions, respectively $\{\mathrm{VP}, \mathrm{inf}, \mathrm{obj}\}$ and $\{\mathrm{cl}, \mathrm{suj}\}$ in this instance, and refine our grammar with the following formula:

$$
\begin{aligned}
[\downarrow^*]\mathit{demande} \supset \big(& \langle(\uparrow;\uparrow;\rightarrow^+) + (\uparrow;\leftarrow^+;\mathrm{cl?})\rangle \mathrm{obj} && \text{(at least one object)} \\
& \wedge \langle(\uparrow;\uparrow;\leftarrow) + (\uparrow;\leftarrow^+;\mathrm{cl?})\rangle \mathrm{suj} && \text{(at least one subject)} \\
& \wedge \bigwedge_{f \in \{\mathrm{suj},\mathrm{obj},\mathrm{aobj}\}} \langle\uparrow;\leftarrow^+;\mathrm{cl?}\rangle f \supset \neg\langle\uparrow;\uparrow;(\leftarrow + \rightarrow^+)\rangle f \big) \\
& && \text{(a clitic argument forbids the corresponding canonical argument)}
\end{aligned}
$$

Interestingly, such $\mathrm{PDL}_{\mathsf{tree}}$ constraints can easily be tested against tree corpora to check their validity; see (Lai and Bird, 2010) on using $\mathrm{PDL}_{\mathsf{tree}}$-like query languages to this end. We checked that the above $\mathrm{PDL}_{\mathsf{tree}}$ formula was satisfied by the trees in the Sequoia treebank (Candito and Seddah, 2012) (using an XPath processor).

Discussion. In this approach, the CFG can be a very permissive, over-generating one, like the probabilistic grammars extracted from treebanks,[1] since it is later refined by the $\mathrm{PDL}_{\mathsf{tree}}$ constraints. We are not aware of any linguistic rationale for cycles in CFGs; on the other hand, ε-productions are sometimes used as placeholders for *moved constituents*. However, in such analyses, the moved constituent and the placeholder are *coindexed*, i.e. related through an additional relation, which

- requires a richer class of models than mere trees over a finite alphabet if we want to make the coindexation explicit (see Figure 1b for an example), and
- can be simulated by a $\mathrm{PDL}_{\mathsf{tree}}$ formula, as seen with the connection we establish between a clitic and the corresponding missing argument in Example 4.

We therefore expect our grammars to be both acyclic and ε-free—and we could check that this was indeed the case on the three rather different CFGs proposed by Moore (2004) for natural language parsing benchmarks.

On the logical side, it seems necessary to be able to use e.g. general successors (recall Example 3). Palm (1999) and Lai and Bird (2010) argue that $\mathrm{PDL}_{\mathsf{cp}}$ provides an appropriate expressiveness for linguistic queries.

3.2 Application: Ambiguity Filtering

Ambiguities in context-free grammars describing the syntax of programming languages are a severe issue, as they might lead to different semantic interpretations, and complicate the use of deterministic parsers—they basically require manual fiddling. They are also quite useful, as they allow for more concise and more readable grammars, and it is actually uncommon to find a language reference proposing an unambiguous grammar.

A nice way of dealing with ambiguities at parse time is to build a parse forest and *filter out* the unwanted trees (Klint and Visser, 1994). In contrast with tinkering with parsers, this allows to implement the "side constraints" of language references as declarative rules, which, beyond readability and maintainability concerns (Kats et al., 2010), also enables some amount of static reasoning and optimization.

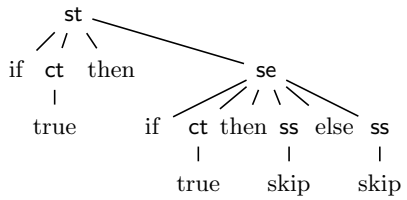
Example 5 (Dangling Else)**.** We propose to use $\mathrm{PDL}_{\mathsf{tree}}$ formulæ to filter out unwanted parses. Consider the following regular tree grammar for statements:[2]

$$S \to \mathsf{st}(\text{if } C \text{ then } S) \mid \mathsf{se}(\text{if } C \text{ then } S \text{ else } S) \mid \mathsf{sw}(\text{while } C\ S) \mid \mathsf{ss}(\text{skip})$$
$$C \to \mathsf{ct}(\text{true}) \mid \mathsf{cf}(\text{false})$$

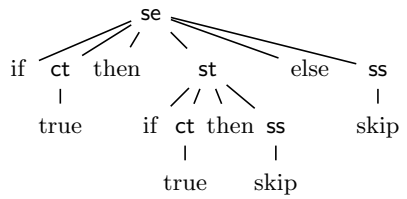
Feeding this grammar to a LALR(1) parser generator like GNU/bison, we find a single shift/reduce conflict, where the parser has a choice on inputs like "if true then if true then skip else skip", upon reaching the "else" symbol, between

[1] Moore (2004) finds an average of 7.2×10^{27} different parse trees per sentence with a grammar extracted from the Penn treebank!

[2] We use a regular tree grammar in a restricted way to label internal nodes differently depending on the chosen production; this allows for a simpler $\mathrm{PDL}_{\mathsf{tree}}$ formula but has otherwise no impact as the language remains local.



(a) Parse when preferring shift over reduce.



(b) Parse when preferring reduce over shift.

Figure 2: Two parses for the ambiguous input "if true then if true then skip else skip" with the grammar of Example 5.

reading further (Figure 2a), and reducing first and leaving this else for later (Figure 2b). The usual convention in programming languages is a greedy one, where shift is always chosen. However, disambiguation by choosing between shift or reduce parsing actions is error-prone, and there are cases where both alternatives are incorrect on some inputs (see (Schmitz, 2010) for an example in Standard ML).

A $\mathrm{PDL}_{\mathsf{tree}}$ formula that accepts the desired tree of Figure 2a but rejects the one of Figure 2b should check that no "else" node can be a general successor (in the sense of Example 3) of an "st" node:

$$\neg\langle\downarrow^*\rangle(\mathsf{st} \wedge \langle\prec\rangle\mathrm{else})\,.$$

Observe that a general successor path $\prec$ is really needed here, because the "st" node can be at the end of an arbitrarily long sequence of "sw" nodes from nested "while" statements.

A very similar approach was proposed by Thorup (1994), who used simple tree patterns for similar purposes. Both tree patterns and $\mathrm{PDL}_{\mathsf{tree}}$ formulæ can be compiled into the grammar, so that only the desired trees can be generated, allowing to use deterministic parsers or ambiguity checking tools (Schmitz, 2010). $\mathrm{PDL}_{\mathsf{tree}}$ formulæ are strictly more expressive than patterns; the dangling else example required an involved extension of patterns in (Thorup, 1996).

Discussion. The grammars used for programming languages are always acyclic—tools like GNU/bison will detect and reject cyclic grammars—but ε-productions are fairly common.

On the logic side, $\mathrm{PDL}_{\mathsf{cp}}$ is required in order to express general successor paths as in Example 5. This seems expressive enough for most tasks, but *layout sensitive* syntax would be beyond its grasp: in programming languages like Haskell or Python, the indentation level is used to delimit statement blocks—

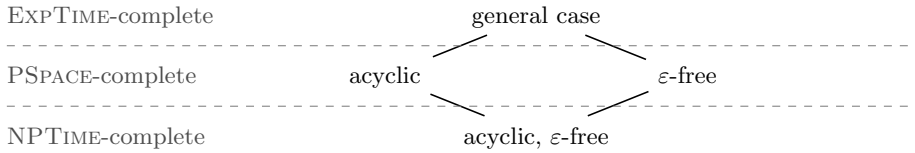


Figure 3: The complexity of the PFMC problem, depending on the grammar characteristics.

differentiating between possible parses then requires some limited counting capabilities, or at least infinite label sets.

Excluding a tree considered individually is one approach among others to ambiguity filtering (Klint and Visser, 1994). A popular alternative considers the parse forest, i.e. the tree automaton $\mathcal{A}_{G,w}$ itself. The ambiguity resolution of Example 5 on the input "if true then if true then skip else skip" can be simply stated as a preference $\mathsf{st} > \mathsf{se}$ implying that the rule

$$(0, S, 9) \to \mathsf{st}\big((0, \text{if}, 1)(1, C, 2)(2, \text{then}, 3)(3, S, 9)\big)$$

is preferred over the rule

$$(0, S, 9) \to \mathsf{se}\big((0, \text{if}, 1)(1, C, 2)(2, \text{then}, 3)(3, S, 7)(7, \text{else}, 8)(8, S, 9)\big)$$

in the automaton $\mathcal{A}_{G,w}$. Such disambiguation rules are easy to write, but they are also inherently *dynamic*: they cannot be compiled into the grammar, because whether the rule will be triggered depends on whether an ambiguity appears there—an undecidable problem.

4 Complexity Results

We investigate in this section the complexity of the parse forest model-checking problem. We obtain a classification of complexities depending on the properties of the grammar (see Figure 3). Interestingly, our hardness results always hold for a formula φ in the rather restricted fragment $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$, and generally hold already for fixed G and/or w. These bounds use logarithmic space reductions.

Turning first to the complexity in the general case, an immediate consequence of classical results in the field (e.g. Calvanese et al., 2009, Theorem 7) is that it lies in ExpTime.

Proposition 1. *PFMC is in* ExpTime.

Proof Sketch. We can assume G to be in quadratic form and φ to work on binary trees that encode unranked trees with the $\swarrow$ and $\to$ relations, as these transformations only incur a linear cost. Then, construct the tree automaton $\mathcal{A}_{G,w}$ of size $O(|G| \cdot |w|^3)$ that recognizes the set of parse trees of w in G and the tree automaton $\mathcal{A}_\varphi$ of size $2^{p(|\varphi|)}$ for a polynomial p that recognizes the models of φ: it suffices to test the emptiness of their product automaton, which can be performed in time linear in $|G| \cdot |w| \cdot 2^{p(|\varphi|)}$ for a polynomial p. □

An interesting consequence of the proof of Proposition 1 is that the PFMC problem is PTime-complete when the $\mathrm{PDL}_{\mathsf{tree}}$ formula is fixed, pleading for using small formulæ in practice.

Our proof for Proposition 1 does not benefit from the specificities of the PFMC problem: any satisfiability problem in presence of a tree language would use the same algorithm. Therefore, we might still hope for the existence of a more efficient solution, but adapting the proof of ExpTime-hardness for PDL satisfiability from (Blackburn et al., 2001), we obtain:

Proposition 2. *PFMC is* ExpTime*-hard, even for fixed G and w and for φ in* $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$.

Proof Idea. We reduce from the *two-players corridor tiling game* of Chlebus (1986). We fix $w = \varepsilon$ and also fix G to generate a parse forest encoding game trees; we use a $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ formula φ to check that there exists a winning strategy. See Appendix A for details. □

As can be seen from this proof idea, the fact that $w = \varepsilon$ and G is cyclic plays an important role, because the parse forest is essentially unconstrained. This is a good incentive to examine what happens when G is acyclic and/or ε-free, especially since those cases are most relevant for the applications we described in Section 3.

4.1 The Acyclic ε-free Case: Mixed Model-Theoretic Syntax

Let us therefore consider the other end of our spectrum, which we claimed was of particular relevance for the mixed approach to model-theoretic syntax we presented in Section 3.1: if G is acyclic and ε-free, then $\mathcal{A}_{G,w}$ is a non-recursive tree automaton generating a *finite* parse forest, albeit it might contain exponentially many trees. This yields an ExpTime algorithm that performs $\mathrm{PDL}_{\mathsf{tree}}$ model-checking (in Ptime (Lange, 2006)) on each tree individually. We can try to refine this first approach and resort to (Benedikt et al., 2008, Lemma 7.5), which entails that the problem for the $\mathrm{PDL}_{\mathsf{core}}$ fragment is in PSpace, but we can do a bit better:

Proposition 3. *PFMC with acyclic and ε-free grammars is* NPTime*-complete; hardness holds even for fixed G and for φ in* $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$.

Proof Idea for the Upper Bound. We show that the parse trees in $L(\mathcal{A}_{G,w})$ are of polynomial size in $|G|$ and $|w|$, hence we can nondeterministically guess such a tree and check that it is a model of φ in polynomial time. See Appendix B.1 for details. □

Proof Idea for the Lower Bound. We reduce from 3SAT with a fixed grammar G and a $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ formula φ; see Appendix B.2 for details. □

4.2 Non-Recursive DTDs

Let us turn now to the more involved cases where G is either acyclic or ε-free: we rely in both cases for the upper bounds on the same result that extends Lemma 7.5 of Benedikt et al. (2008) to handle $\mathrm{PDL}_{\mathsf{tree}}$ instead of $\mathrm{PDL}_{\mathsf{core}}$:

Proposition 4. *Satisfiability of* $\mathrm{PDL}_{\mathsf{tree}}$ *in presence of a non-recursive DTD is* PSpace*-complete.*

Let us recall that a *document type definition* (DTD) is a generalized CFG $D = \langle N, P, S\rangle$ where P is a mapping from N to *content models* in $\mathrm{Reg}(N^*)$ the set of regular languages over N—we will assume these content models to be described by finite automata (NFA). Given D, the derivation relation $\Rightarrow$ relates $\beta A\gamma$ to $\beta\alpha\gamma$ iff α is in $P(A)$; a DTD is *non-recursive* if no nonterminal has a derivation $A \Rightarrow^+ \beta A\gamma$ for some β, γ in N^*. Note that a non-recursive DTD might still generate an infinite tree language, but that all its trees will have a depth bounded by $|N|$.

Proof Idea for Proposition 4. The hardness part is proven by Benedikt et al. (2008) in their Proposition 5.1.

For the upper bound, we reduce to the emptiness problem of a 2-way alternating parity *word* automaton, which is in PSPACE (Serre, 2006). The key idea, found in Benedikt et al.'s work, is to encode trees of bounded depth as XML strings (i.e. with opening and closing tags): both the DTD D and the formula φ can then be encoded as alternating parity word automata $\mathcal{A}_D$ and $\mathcal{A}_\varphi$ of polynomial size. Our construction for $\mathcal{A}_\varphi$ is a bit different from that of Benedikt et al.; for instance, we cannot assume it to be loop-free. See Appendix C for details. □

4.2.1 Acyclic Case: Ambiguity Filtering

We are now ready to attack the case of acyclic grammars. This restriction is enough to ensure that the parse forest is finite, and, more importantly, $\mathcal{A}_{G,w}$ is trivially non-recursive, thus Proposition 4 immediately yields an PSPACE upper bound. In fact, this is optimal:

Proposition 5. *PFMC with acyclic grammars is* PSPACE*-complete; hardness holds even for fixed w and for φ in* $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$.

Proof Sketch. Because G is acyclic, for any w, the trimmed version of $\mathcal{A}_{G,w}$ is a non-recursive tree automaton. Indeed, in this automaton, if a state (i_0, A, i_k) of $\mathcal{A}_{G,w}$ rewrites in n steps into a tree t with leaves labeled by $(i_0, X_1, i_1)\cdots(i_{k-1}, X_k, i_k)$, then $A \Rightarrow^n X_1 \cdots X_k$ in G. If the automaton is trim, then the existence of a state (i, B, j) implies that B derives the factor $a_{i+1}\cdots a_j$ of w. Thus, if (i, A, j) were to rewrite in at least one step into a tree $C[(i, A, j)]$, then there would be a cycle $A \Rightarrow^+ A$ in G, a contradiction. It remains to relabel the rules $(i, A, j) \to A(q_1 \cdots q_m)$ of $\mathcal{A}_{G,w}$ to $(i, A, j) \to (i, A, j)(q_1 \cdots q_m)$ to obtain a *local* non-recursive tree automaton, which is just a particular case of a non-recursive DTD, and interpret the propositions p in $V' = \mathsf{AP}$ as $\bigvee_{0\le i\le j\le n}(i, p, j)$ over $Q_{G,w}$ in φ to apply Proposition 4 and obtain the upper bound.

The lower bound holds for the $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ satisfiability problem in presence of non-recursive and no-star DTDs (Benedikt et al., 2008, Proposition 5.1), which is easy to reduce to our problem by simply adding ε-leaves in the DTD; this lower bound thus already holds for a fixed $w = \varepsilon$. □

4.2.2 ε-Free Case: $\mathrm{PDL}_{\mathsf{tree}}$ Recognition

We reach the last case of our study. Due to cycles, an ε-free grammar G can have infinitely many parses for a given input string w, and its parse trees unbounded depth. Nevertheless, recursions in a parse forest of an ε-free grammar display a

particular shape: they are *chains* of unit rules $q_i \to A_i(q_{i+1})$. The key idea here is that such chains define regular languages of single-strand branches, which can be encoded in a non-recursive DTD by "rotating" them, i.e. seeing the chain as a siblings sequence instead of a parents sequence, taking advantage of the DTD's ability to describe trees of unbounded rank.

Proposition 6. *PFMC with ε-free grammars is in* PSPACE.

Proof Idea. The algorithm starts by constructing $\mathcal{A}_{G,w}$ in polynomial time on binarized trees; we want to reduce the problem to the satisfiability problem for $\text{PDL}_{\mathsf{tree}}$ in presence of a non-recursive DTD and use Proposition 4. As in the proof of Proposition 5, we consider "localized" rules $q \to q(q_1\, q_2)$ of $\mathcal{A}_{G,w}$, and replace them by productions of the form $q \to \mathsf{chains}(q_1)\mathsf{chains}(q_2)$ where the $\mathsf{chains}(q_i)$ are the languages of single chains out of q_i. By suitably labeling our trees, we can interpret φ over those transformed trees. See Appendix D.1 for details. □

Proposition 6 is optimal:

Proposition 7. *PFMC with ε-free grammars is* PSPACE*-hard, even for fixed G and w and for φ in* $\text{PDL}_{\mathsf{core}}[\downarrow]$.

Proof Idea. The proof is by reduction from membership in a linear bounded automaton. We fix $w = a$ for some symbol a of Σ, and also fix the CFG G to basically generate any single-strand tree with a root S and a leaf a over a fixed alphabet. A $\text{PDL}_{\mathsf{core}}[\downarrow]$ formula of polynomial size then checks that this tree encodes an accepting run of the LBA. See Appendix D.2 for details. □

$\text{PDL}_{\mathsf{tree}}$ Recognition. A key question if model-theoretic syntax is to be used in practice for natural language processing is the following *recognition problem*:

$\text{PDL}_{\mathsf{tree}}$ Recognition Problem.

input a $\text{PDL}_{\mathsf{tree}}$ formula φ, a word w in AP^*, and a distinguished proposition s in AP,

question does there exist a tree t with yield w and root label s s.t. $t \models \varphi$? □

Note in particular that the statement of the problem excludes ε-labeled leaves, which would require a different formulation and would yield an EXPTIME-complete problem.

The previous approaches to the recognition problem have used tree automata techniques (e.g. Cornell, 2000) or tableau-like techniques (Palm, 2004). In both cases, exponential time upper bounds were reported by the authors—to be fair, these algorithms solve the *parsing* problem and find a representation of *all* the parses for w compatible with φ—, but we can improve on this thanks to Proposition 6:

Corollary 1. $\text{PDL}_{\mathsf{tree}}$ *recognition is* PSPACE*-complete; hardness holds even for fixed w and for φ in* $\text{PDL}_{\mathsf{core}}[\downarrow]$.

Proof Sketch. The lower bound stems from an easy reduction from Proposition 7: we can encode the grammar G into a $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ formula φ_G as in Example 2 and reduce to the parsing problem for w and $\varphi \wedge \varphi_G$.

For the upper bound, we can assume as usual φ to work on a binary encoding of trees. The idea is to reduce to the PFMC problem with a "universal" CFG that accepts all the trees of rank at most 2 over AP. A smallish issue is that we need to separate between nonterminal and terminal labels, but we can create a disjoint copy $N \stackrel{\text{def}}{=} \{P \mid p \in \mathsf{AP}\}$ of AP and interpret φ as a formula over $N \uplus \mathsf{AP}$ with $P \vee p$ as the interpretation of p. This grammar has then S as axiom and productions $A \to X\ Y$ and $A \to X$ for all A in N and X, Y in V, and we can resort to Proposition 6 to conclude. □

5 Conclusion

Because $\mathrm{PDL}_{\mathsf{tree}}$ formulæ can freely navigate in trees, properties that rely on long-distance relations are convenient to express, in contrast with the higly local view provided by a grammar production. However, this expressiveness comes at a steep price, as complexity problems on $\mathrm{PDL}_{\mathsf{tree}}$ are typically ExpTime-complete instead of PTime-complete on CFGs.

The $\mathrm{PDL}_{\mathsf{tree}}$ model-checking of the parse trees of a CFG allows to mix the two approaches, using a grammar for the bulk work of describing trees and using more sparingly a $\mathrm{PDL}_{\mathsf{tree}}$ formula for the fine work. We argue that this trade-off would find natural applications in computational linguistics and compilers construction, where sensible restrictions on the grammar lower the complexity to NPTime or PSpace.

An additional consequence is that the recognition problem for $\mathrm{PDL}_{\mathsf{tree}}$ is in PSpace. This is a central problem in model-theoretic syntax, and this lower complexity suggests that "lazy" approaches, in the spirit of the tableau construction of Palm (2004), should perform significantly better than the automata constructions of Cornell (2000).

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A General Case

PDL satisfiability is known to be ExpTime-complete in general (Fischer and Ladner, 1979). The general case of the parse forest model-checking problem, i.e. when G is an arbitrary grammar, is also ExpTime-complete. The upper bound follows from classical techniques for the upper bound (Calvanese et al., 2009)—see the proof sketch of Proposition 1.

The lower bound could be proven by a reduction from PDL satisfiability using a "universal" CFG as in the proof of Corollary 1. However, this proof does not lend itself very easily to the restricted case we want to consider, where w and G are fixed and φ is a downward $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ formula. We present in this section a reduction from the *two-player corridor game*, which is known to be ExpTime-hard (Chlebus, 1986), adapted from a similar proof for the hardness of PDL satisfiability by Blackburn et al. (2001, Theorem 6.52).

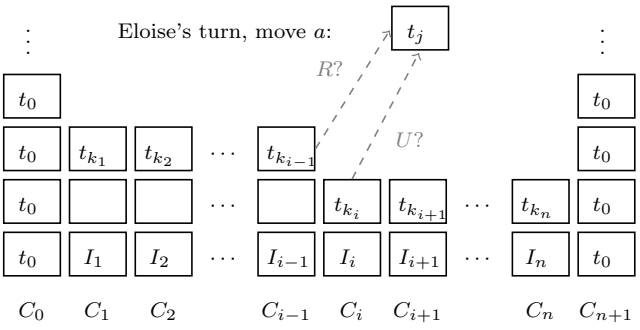
Two Player Corridor Game sees two players, Eloise and Abelard, compete by tiling a corridor. The tiles are squares decorated by $s+2$ different patterns $T = \{t_0, \ldots, t_{s+1}\}$; two binary relations U and R over T tell if a tile can be placed on top of the other and to the right of the other. Two tiles are distinguished: t_0 is called the *white* tile and t_{s+1} the *winning* tile. The corridor is made of $n+2$ columns of infinite height, with the first and last columns filled with white tiles t_0 and delimiting n columns for the play. The initial bottom row is tiled by a sequence $I_1 \cdots I_n$ of tiles, which is assumed to be correct, i.e. to respect the R relation.

The players alternate and choose a next tile in T and place it in the next position, which is the lowest leftmost free one—thus the chosen tile should match the tile to its left (using R) and the tile below (using U)—; see Figure 4a. Eloise starts the game and wins if after a finite number of rounds, the winning tile t_{s+1} is put in column 1. Given an instance of the 2-players corridor tiling game, i.e. $\langle s+2, I_1 \cdots I_n, R, U\rangle$, deciding whether Eloise has a *winning strategy*, i.e. a way of winning no matter what Abelard plays, is ExpTime-complete.

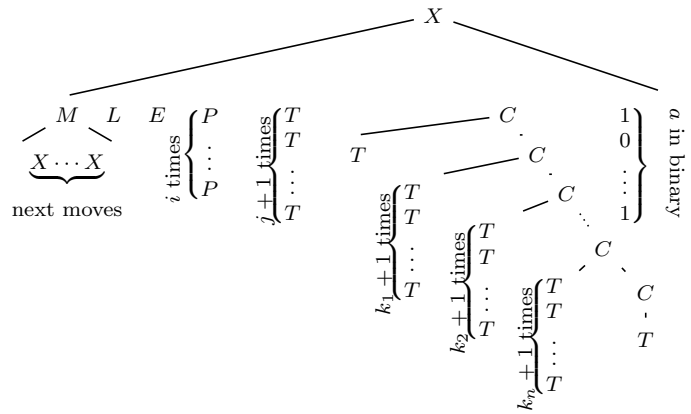
Notation. We represent strategy trees as parse trees. Our $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ formula φ will ensure that the parse tree is indeed a valid game tree, and that it encodes a winning strategy for Eloise.

A game turn is encoded locally by an X-labeled node and its immediate children, with the next reachable configurations reachable through a path of M-labeled nodes. More precisely, each X node has the following children (see Figure 4b):

- a node labeled either W or L, stating that the configuration is winning or not for Eloise,
- a node labeled either E or A, stating whether it is Eloise's or Abelard's turn to move,
- a chain of i P-labeled nodes, stating that the current playing column is C_i,
- a chain of $j+1$ T-labeled nodes, stating that the chosen tile at this turn is t_j,



(a) One of Eloise's turns in the game.



(b) The tree encoding of the turn.

Figure 4: A turn of the 2-players corridor game and its tree encoding.

- a comb-shaped subtree of $n+2$ nodes labeled by C, describing the contents of the top layer in the corridor (which in general spans two rows), with a strand of $k+1$ T-labeled nodes telling for each column that tile t_k is on top,
- a chain of m nodes labeled 0 or 1, encoding in binary the number of the moves made so far; this chain does not need to be longer than $m \stackrel{\text{def}}{=} \lceil \log(2n^{s+3}) \rceil$—or some move would have been repeated.

The Grammar. We *fix* $w \stackrel{\text{def}}{=} \varepsilon$ and $G \stackrel{\text{def}}{=} \langle N, \emptyset, P, X \rangle$ over the nonterminal alphabet $N = \{X, M, W, L, E, A, P, T, C, 0, 1\}$ with productions (with a slightly extended syntax with alternatives built-in the productions right-hand sides):

$$X \to M\,(W \mid L)\,(E \mid A)\,P\,T\,C\,(0 \mid 1) \qquad M \to X \mid X\,M \mid \varepsilon \qquad C \to T \mid T\,C$$
$$P \to P \mid \varepsilon \qquad T \to T \mid \varepsilon \qquad W \to \varepsilon$$
$$L \to \varepsilon \qquad E \to \varepsilon \qquad A \to \varepsilon$$
$$0 \to 0 \mid 1 \mid \varepsilon \qquad 1 \to 0 \mid 1 \mid \varepsilon$$

Observe that all the tree encodings of strategies are generated by G, but that not all the trees of G encode a strategy: for instance, the number of C's might

be different from $n+2$, or the described tiling might not respect the placement constraints, etc. The formula will check these conditions.

Game Structure and Mechanics. We use the non-terminal labels and ε as atomic propositions in our $\mathrm{PDL}_{\mathsf{tree}}$ formula. Because there are at most $s+2$ choices of tiles at each turn, we can define the path

$$\mathsf{move} \stackrel{\text{def}}{=} \sum_{i=1}^{s+2} (\downarrow; M?)^i; \downarrow; X?$$

that relates two successive configurations. Further define the following formulæ for $0 \leq i \leq n+1$, $0 \leq j \leq s+1$, $1 \leq a \leq m$ and b in $\{0,1\}$:

$$\mathsf{p}(i) \stackrel{\text{def}}{=} \langle (\downarrow; P?)^i; \downarrow \rangle \varepsilon \qquad \mathsf{t}(j) \stackrel{\text{def}}{=} \langle (\downarrow; T?)^{j+1}; \downarrow \rangle \varepsilon$$
$$\mathsf{q}(a,b) \stackrel{\text{def}}{=} \langle (\downarrow; (0+1)?)^a \rangle b \qquad \mathsf{c}(i,j) \stackrel{\text{def}}{=} \langle (\downarrow; C?)^{i+1}; (\downarrow; T?)^{j+1}; \downarrow \rangle \varepsilon$$

In an X node, $\mathsf{p}(i)$ holds if the current column is C_i, $\mathsf{t}(j)$ if the chosen tile is T_j, $\mathsf{c}(i,k_i)$ if tile T_{k_i} is on top of column C_i, and $\mathsf{q}(m,b)$ if the binary encoding of the current move number has bit a set to b.

We ensure some preliminary structure on the game tree: At the start of the play, the current player must be Eloise, and the referee should have placed the initial tiles in the first row. The counter must be initialized to zero.

$$\varphi_1 \stackrel{\text{def}}{=} X \wedge (\langle\downarrow\rangle E) \wedge \mathsf{p}(1) \wedge \bigwedge_{i=1}^{n} \mathsf{c}(i, I_i) \wedge \bigwedge_{a=1}^{m} \mathsf{q}(a,0) \ .$$

In every state, the tiles in columns 0 and $n+1$ should be the white tile t_0.

$$\varphi_2 \stackrel{\text{def}}{=} [\downarrow^*] X \supset \mathsf{c}(0,0) \wedge \mathsf{c}(n+1,0) \ .$$

In every state with current column i, the next move should be at position $(i \bmod n) + 1$.

$$\varphi_3 \stackrel{\text{def}}{=} \bigwedge_{i=1}^{n} [\downarrow^*](X \wedge \mathsf{p}(i)) \supset [\mathsf{move}]\mathsf{p}((i \bmod n) + 1) \ .$$

The columns are updated with the correct tiles.

$$\varphi_4 \stackrel{\text{def}}{=} \bigwedge_{i=1}^{n} \bigwedge_{j=0}^{s+1} [\downarrow^*](X \wedge \mathsf{p}(i) \wedge \mathsf{t}(j)) \supset [\mathsf{move}]\mathsf{c}(i,j)$$
$$\wedge \bigwedge_{i \neq j=1}^{n} \bigwedge_{k=0}^{s+1} (X \wedge \mathsf{p}(i) \wedge \mathsf{c}(j,k)) \supset [\mathsf{move}]\mathsf{c}(j,k) \ .$$

Players alternate.

$$\varphi_5 \stackrel{\text{def}}{=} [\downarrow^*](X \wedge \langle\downarrow\rangle E \equiv [\mathsf{move}]\langle\downarrow\rangle A) \ .$$

The chosen tiles verify the adjacency constraints: define for this the Boolean:

$$\mathsf{adj}(i,j,k,\ell) \stackrel{\text{def}}{=} t_\ell \, U \, t_j \wedge (i > 0 \supset t_k \, R \, t_j) \wedge (i = 0 \supset t_0 \, R \, t_j) \wedge (i = n \supset t_j \, R \, t_0)$$

$$\varphi_6 \stackrel{\text{def}}{=} \bigwedge_{i=1}^{n} \bigwedge_{j,k,\ell=0}^{s+1} [\downarrow^*](X \wedge \mathsf{p}(i) \wedge \mathsf{t}(j) \wedge \mathsf{c}(i-1,k) \wedge \mathsf{c}(i,\ell)) \supset \mathsf{adj}(i,j,k,\ell) \; .$$

The counter is incremented.

$$\varphi_7 \stackrel{\text{def}}{=} \bigwedge_{d=1}^{m} \bigwedge_{a=1}^{d-1} \bigwedge_{b \in \{0,1\}} [\downarrow^*]\big(X \wedge \mathsf{q}(a,b) \wedge \mathsf{q}(d,0) \wedge \bigwedge_{e=d+1}^{m} \mathsf{q}(e,1)\big)$$
$$\supset [\mathsf{move}]\big(\mathsf{q}(a,b) \wedge \mathsf{q}(d,1) \wedge \bigwedge_{e=d+1}^{m} \mathsf{q}(e,0)\big) \; .$$

Winning Strategy. The previous formulæ were making sure that the tree would be a proper game tree. We want now to check that it describes a winning strategy for Eloise: We should check that all the possible moves of Abelard are tested:

$$\varphi_8 \stackrel{\text{def}}{=} \bigwedge_{i=1}^{n} \bigwedge_{j,k,\ell=0}^{s+1} [\downarrow^*](X \wedge \mathsf{p}(i) \wedge \langle\downarrow\rangle E \wedge \mathsf{t}(k) \wedge \mathsf{c}((i \bmod n)+1,\ell) \wedge \mathsf{adj}(i,j,k,\ell)) \supset \langle\mathsf{move}\rangle \mathsf{t}(j) \; .$$

Finally, the winning condition should be met:

$$\varphi_9 \stackrel{\text{def}}{=} (\langle\downarrow\rangle W) \wedge [\downarrow^*](X \wedge \langle\downarrow\rangle W) \supset \big(\mathsf{c}(1,s+1)$$
(the game is immediately winning)
$$\vee \, ((\langle\downarrow\rangle E) \wedge (\langle\mathsf{move};\downarrow\rangle W))$$
(Eloise can win later)
$$\vee \, (\langle\downarrow\rangle A) \wedge (\langle\mathsf{move}\rangle \top) \wedge [\mathsf{move}]\langle\downarrow\rangle W)\big) \; .$$
(None of Abelard's moves can prevent Eloise from winning)

Finally, our final $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ formula is $\varphi \stackrel{\text{def}}{=} \bigwedge_{i=1}^{9} \varphi_i$. Because G and w are fixed and φ can be computed in space logarithmic in the size of the game instance, we have therefore shown the general PFMC problem to be ExpTime-hard.

B Acyclic and ε-Free Case: Proposition 3

We prove here Proposition 3: the PFMC problem is NPTime-complete for acyclic and ε-free grammars.

B.1 Upper Bound

Proof. Let us show that the parse trees in $L_{G,w}$ are of polynomially bounded size. The NPTime algorithm then guesses a tree in $L_{G,w}$ and checks that it is a model in polynomial time (Lange, 2006).

Claim 9. Let $G = \langle N, \Sigma, P, S\rangle$ be an acyclic and ε-free CFG. Let $w \in \Sigma^*$. Any parse tree t in $L_{G,w}$ has at most $|N|(|w|-1)+|w|$ nodes.

Consider the run of $A_{G,w}$ on t: each node of t is labeled by a state (i, A, j) describing two positions $0 \leq i \leq j \leq n$ in w and a nonterminal A in N. Because G is ε-free, we know that $i < j$. We claim that the set of nodes labelled with positions (i, j) forms a connected chain.

To see this, suppose two nodes a and b are both labelled with positions (i, j). Suppose first that neither a nor b is an ancestor of the other. Let then $c \notin \{a, b\}$ be their least common ancestor (lca): c must have at least two children, and its children will be labelled with non-overlapping positions—recall that $i < j$. Only one of these non-overlapping intervals can contain the interval (i, j). The child corresponding to that interval would then be the lca of a and b, in contradiction with c being their lca: hence one of a or b is the lca of a and b.

Suppose now wlog. that a is an ancestor of b. Observe that a descendant of a would be labelled with a sub-interval of (i, j), and an ancestor of b would be labelled with a super-interval of (i, j). This forces every node in the path from a to b to be labelled with (i, j). Hence, the nodes labelled with (i, j) form a connected chain.

Since G is acyclic, each chain of nodes $(i, A_1, j), (i, A_2, j) \cdots (i, A_p, j)$ having the same positions (i, j) cannot have a non-terminal A_k occuring twice, or the grammar would allow a cycle. Therefore, each such chain will have at most $|N|$ nodes. We can "collapse" these chains to form a tree where each (i, j) pair appears at most once, and every node (except the leaves) has at least two children. Since there are exactly $|w|$ leaves (G is ε-free), there can be at most $|w| - 1$ internal nodes in such a tree. We obtain that there were at most $|N|(|w|-1)$ internal nodes in the original parse tree, i.e. at most $|N|(|w|-1)+|w|$ nodes in the full parse tree. □

B.2 Lower Bound

Proof. We reduce 3SAT to our problem.

Fix the grammar $G \stackrel{\text{def}}{=} \langle \{S, F, T\}, \{a\}, P, S\rangle$ with productions:

$$S \to S\,F \mid S\,T \mid F \mid T \qquad F \to a \qquad T \to a$$

and consider an instance $\psi = \bigwedge_{i=1}^{m} C_i$ of 3SAT where each C_i is a disjunction of literals over n variables $\{x_1, \ldots, x_n\}$. Define $w \stackrel{\text{def}}{=} a^n$.

Any parse tree t of w will have a "comb" shape of length n with S-labeled nodes, each giving rise to one of F or T as a child. The parse forest is thus in bijection with the set of valuations of $\{x_1, \ldots, x_n\}$: if the value of variable x_i is 0, then in our encoding, the ith S node has a node with label F as a child; otherwise, it has a node with label T as a child.

Given such an encoded valuation, our formula φ must verify that each clause is satisfied. For a clause $C_i = \ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3}$ with $\ell_{i,j} = x_{k_j}$ or $\ell_{i,j} = \neg x_{k_j}$, define $\varphi_i \stackrel{\text{def}}{=} \bigvee_{j=1}^{3} \langle (S; \to)^{k_j} \rangle \beta_{i,j}$ where $\beta_{i,j} = F$ if $\ell_{i,j} = \neg x_{k_j}$ and $\beta_{i,j} = T$ otherwise. Finally, let $\varphi \stackrel{\text{def}}{=} \bigwedge_{i=1}^{m} \varphi_i$. Then $t \models \varphi$ if and only if the corresponding assignment of the variables is a satisfying assignment. Because G is fixed and w and φ can be computed in space logarithmic in the size of the 3SAT instance, this shows the NPTime-hardness of the PFMC problem in the acyclic ε-free case. □

C Model-Checking Non-Recursive DTDs: Proposition 4

We present in this section a proof of Proposition 4: the satisfiability of a $\mathrm{PDL}_{\mathsf{tree}}$ formula φ in presence of a non-recursive DTD D is PSPACE-complete.

The lower bound is proved as Proposition 5.1 by Benedikt et al. (2008), and we follow their general proof plan from Lemma 7.5 for the upper bound. As presented in the main text, the fact that we consider a non-recursive DTD means that the height of any tree of interest is bounded by $|N|$ the number of nonterminals of the DTD. The proof plan is then to consider XML word encodings of trees, and construct two *2-way alternating parity word automata* (2APWA) $\mathcal{A}_D$ and $\mathcal{A}_\varphi$ of polynomial size which will respectively recognize the XML encodings of the trees of D and of the models of φ of height bounded by $|N|$. Then, by taking the conjunction of the two automata, we reduce the initial satisfiability problem to a 2APWA emptiness problem, which is known to be in PSPACE by the results of Serre (2006).

We can find a suitable construction for an automaton $\mathcal{A}_D$ for D as Claim 7.7 of (Benedikt et al., 2008), thus we will only present the construction of $\mathcal{A}_D$.

XML Encoding. Define the alphabet

$$\mathsf{XML}(N) \stackrel{\text{def}}{=} \{\langle X\rangle, \langle /X\rangle \mid X \in N\}$$

and choose a fresh root symbol r not in N. We encode our a tree t as $\langle \mathsf{r}\rangle \mathsf{stream}(t) \langle /\mathsf{r}\rangle$ where the *XML streaming* function is defined inductively on terms by

$$\mathsf{stream}(f(t_1 \cdots t_m)) \stackrel{\text{def}}{=} \langle f\rangle \mathsf{stream}(t_1) \cdots \mathsf{stream}(t_m) \langle /f\rangle \ .$$

2-Way Alternating Parity Word Automata. A *positive boolean formula* f in $\mathbb{B}^+(X)$ over a set X of variables is defined by the syntax

$$f ::= \top \mid \bot \mid f \wedge f \mid f \vee f \ .$$

A subset $X' \subseteq X$ satisfies a formula f, written $X' \models f$, if the formula is satisfied by the valuation $x \mapsto \top$ whenever $x \in X'$ and $x \mapsto \bot$ if $x \in X' \setminus X$.

A *2-way alternating parity word automaton* is a tuple $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, c\rangle$ where Q is a finite set of states, Σ a finite alphabet, $q_0 \in Q$ an initial state, c a coloring from Q to a finite set of priorities $C \subseteq \mathbb{N}$, and δ a transition function from $Q \times \Sigma$ to $\mathbb{B}^+(Q \times \{-1, 0, 1\})$ that associates to a current state and current symbol boolean formulæ on pairs (q', d) of a new state q' and a direction d.

A *run* of a 2APWA on a finite word $w = a_1 \cdots a_n$ in Σ^* is a generally infinite tree with labels in $Q \times \{1, \ldots, n\}$ holding a current state and a current position in w, such that the root is labeled $(q_0, 1)$, and every node labeled (q, i) with has a children set $\{(q_1, i_1), \ldots, (q_m, i_m)\}$ that satisfies $\delta(q, a_i)$. A run is *accepting* iff for every branch, the smallest priority $c(q)$ that occurs infinitely often among the nodes (q, i) is even—this also means in particular that any finite run is accepting—, and w is accepted if there exists some accepting run for it.

Inductive Construction. We construct $\mathcal{A}_\varphi \stackrel{\text{def}}{=} \langle Q_\varphi, \Sigma, \delta_\varphi, q_{0,\varphi}, c_\varphi\rangle$ by induction on the subterms of the formula φ. We work with the alphabet $\Sigma \stackrel{\text{def}}{=} \mathsf{XML}(N) \uplus \{\langle \mathsf{r}\rangle, \langle /\mathsf{r}\rangle\}$ and set $n \stackrel{\text{def}}{=} |N|$—which is the maximum height of any tree of D. The guiding principles in this construction is that our inductively constructed automaton will track their height relative to that of their starting position. Because we are working on trees of bounded depth, this can be achieved by considering states that combine a "control" state with a height in $\{-n, \dots, n\}$.

Let us start with the base cases for node formulæ: by convention, our automata for a node formulæ must check that their starting positions are labeled by opening tags:

$\mathcal{A}_p$ The automaton checks if it starts at an opening node $\langle p\rangle$. It immediately goes into either an accepting or rejecting state. Formally, $Q_p \stackrel{\text{def}}{=} \{q_{p,0}\}$, the coloring c_p maps $q_{p,0}$ to 1, and $\delta(q_{p,0}, \langle p\rangle) \stackrel{\text{def}}{=} \top$ and $\delta_p(q_{p,0}, X) = \bot$ for all $X \neq \langle p\rangle$.

$\mathcal{A}_\top$ The automaton immediately goes into an accepting state, unless it is at a closing node or the root node. Formally, $Q \stackrel{\text{def}}{=} \{q_{\top,0}\}$ and $c_\top$ is defined by $c_\top(q_{\top,0}) \stackrel{\text{def}}{=} 1$; $\delta_\top(q_0, X)$ is defined as $\top$ for $X = \langle p\rangle$ in $\mathsf{XML}(N)$ and as $\bot$ otherwise.

The automata $\mathcal{A}_\pi$ for π a path formula additionally carry a distinguished subset $C_\pi \subseteq Q_\pi$ of *continuation* states, such that there is a "partial run" from some initial position with branches starting from their initial state, which are either infinite but verifying the parity condition, or are finite but end in a continuation state in a position related to the initial one through $[\![\pi]\!]$. Let us see this at work with the base cases of path formulæ:

$\mathcal{A}_\downarrow$ The automaton moves right from the initial node while maintaining the depth relative to this initial node. It stops (goes into a dead state) if it reaches a node at the same or lesser depth than the initial node. All the visited nodes with a relative depth of 1 are direct children of the initial node, and therefore visited by continuation states. We set where $Q_\downarrow \stackrel{\text{def}}{=} \{q_0, q_1, \cdots, q_n\}$ with $q_{\downarrow,0} \stackrel{\text{def}}{=} q_0$; the coloring $c_\downarrow$ is identically 1 on $Q_\downarrow$, $C_\downarrow = \{q_1\}$, and $\delta_\downarrow$ is defined by:

$$\begin{array}{llll}
\delta_\downarrow(q_i, \langle p\rangle) \stackrel{\text{def}}{=} (1, q_{i+1}), & i < n, p \in N & \delta_\downarrow(q_n, \langle p\rangle) \stackrel{\text{def}}{=} \bot, & p \in N \\
\delta_\downarrow(q_i, \langle /p\rangle) \stackrel{\text{def}}{=} (1, q_{i-1}), & i > 1, p \in N & \delta_\downarrow(q_0, \langle /p\rangle) \stackrel{\text{def}}{=} \delta_\downarrow(q_1, \langle /p\rangle) \stackrel{\text{def}}{=} \bot, & p \in N \\
\delta_\downarrow(q_i, \langle /\mathsf{r}\rangle) \stackrel{\text{def}}{=} \bot, & i \in \{0, \dots, n\}\ . & &
\end{array}$$

$\mathcal{A}_\rightarrow$ Similarly to $\mathcal{A}_\downarrow$, the automaton moves right while maintaining the depth relative to the initial node. It fails if it reaches a node at a lesser depth that the initial node. Otherwise, it finds the next node at a same depth as the initial node. Formally, $Q_\rightarrow \stackrel{\text{def}}{=} \{q_0, q_1, \dots, q_n, q_f\}$, $q_{\rightarrow,0} \stackrel{\text{def}}{=} q_0$, the coloring $c_\rightarrow$ is identically 1 on $Q_\downarrow$, there is a unique continuation state

$C_{\rightarrow} \stackrel{\text{def}}{=} \{q_f\}$ when reaching the right sibling, and $\delta_{\rightarrow}$ is defined by

$$\begin{array}{llll}
\delta_{\rightarrow}(q_i, \langle p \rangle) \stackrel{\text{def}}{=} (1, q_{i+1}), & i < n, p \in N & \delta_{\rightarrow}(q_n, \langle p \rangle) \stackrel{\text{def}}{=} \bot, & p \in N \\
\delta_{\rightarrow}(q_i, \langle / p \rangle) \stackrel{\text{def}}{=} (1, q_{i-1}), & i > 1, p \in N & \delta_{\rightarrow}(q_1, \langle / p \rangle) \stackrel{\text{def}}{=} (1, q_f), & p \in N \\
\delta_{\rightarrow}(q_0, \langle / p \rangle) \stackrel{\text{def}}{=} \bot, & p \in N & \delta_{\rightarrow}(q_f, X) \stackrel{\text{def}}{=} \bot, & X \in \Sigma \\
\delta_{\rightarrow}(q_i, \langle / \mathsf{r} \rangle) \stackrel{\text{def}}{=} \bot, & i \in \{0, \ldots, n\} . & &
\end{array}$$

$\mathcal{A}_{\uparrow}$, $\mathcal{A}_{\leftarrow}$ We define these automata similarly to $\mathcal{A}_{\downarrow}$ and $\mathcal{A}_{\rightarrow}$. Observe however that, because we always finish on opening brackets, it is not enough to exchange -1 and 1 in the directions of transitions.

Next, we consider the induction step for path formulæ:

$\mathcal{A}_{\pi_1;\pi_2}$ We combine the automata $\mathcal{A}_{\pi_1}$ and $\mathcal{A}_{\pi_2}$. We add transitions from the continuation states of $\mathcal{A}_{\pi_1}$ at opening nodes to the initial state of $\mathcal{A}_{\pi_2}$. Formally, $Q_{\pi_1;\pi_2} \stackrel{\text{def}}{=} Q_{\pi_1} \uplus Q_{\pi_2}$, $q_{\pi_1;\pi_2} \stackrel{\text{def}}{=} q_{\pi_1,0}$, $c_{\pi_1;\pi_2}$ preserves the priorities of c_{π_1} and c_{π_2}, $C_{\pi_1;\pi_2} \stackrel{\text{def}}{=} C_{\pi_2}$ and $\delta_{\pi_1;\pi_2}$ is defined by

$$\begin{array}{ll}
\delta_{\pi_1;\pi_2}(q_1, X) \stackrel{\text{def}}{=} \delta_{\pi_1}(q_1, X), & q_1 \in Q_{\pi_1} \setminus C_{\pi_1} \\
\delta_{\pi_1;\pi_2}(q_1, \langle p \rangle) \stackrel{\text{def}}{=} \delta_{\pi_1}(q_1, \langle p \rangle) \vee (0, q_{\pi_2,0}), & q_1 \in C_{\pi_1} \\
\delta_{\pi_1;\pi_2}(q_1, \langle / p \rangle) \stackrel{\text{def}}{=} \delta_{\pi_1}(q_1, \langle / p \rangle), & q_1 \in C_{\pi_1} \\
\delta_{\pi_1;\pi_2}(q_2, X) \stackrel{\text{def}}{=} \delta_{\pi_2}(q_2, X), & q_2 \in Q_{\pi_2}
\end{array}$$

for X in Σ and p in N.

$\mathcal{A}_{\pi_1+\pi_2}$ This is a straightforward union: We define $Q_{\pi_1+\pi_2} \stackrel{\text{def}}{=} Q_{\pi_1} \uplus Q_{\pi_2} \uplus \{q_{\pi_1+\pi_2,0}\}$, $C_{\pi_1+\pi_2} \stackrel{\text{def}}{=} C_{\pi_1} \cup C_{\pi_2}$, $c_{\pi_1+\pi_2} \stackrel{\text{def}}{=} c_{\pi_1} \cup c_{\pi_2} \cup \{(q_{\pi_1+\pi_2,0}, 1)\}$, $\delta_{\pi_1+\pi_2} \stackrel{\text{def}}{=} \delta_{\pi_1} \cup \delta_{\pi_2} \cup \{(q_{\pi_1+\pi_2,0}, X, (0, q_{\pi_1,0}) \vee (0, q_{\pi_2,0})) \mid X \in \Sigma\}$.

$\mathcal{A}_{\pi^*}$ This case is similar to that of $\mathcal{A}_{\pi_1;\pi_2}$; we add transitions from the critical states of $\mathcal{A}_{\pi}$ to its own initial state. Define $Q_{\pi^*} \stackrel{\text{def}}{=} Q_{\pi} \uplus \{q_{\pi^*,0}\}$, $C_{\pi^*} \stackrel{\text{def}}{=} \{q_{\pi^*,0}\}$, $c_{\pi^*} \stackrel{\text{def}}{=} c_{\pi} \cup \{(q_{\pi^*,0}, 1)\}$, and

$$\begin{array}{llll}
\delta_{\pi^*}(q_{\pi^*,0}, X) \stackrel{\text{def}}{=} (0, q_{\pi,0}), & & \delta_{\pi^*}(q, \langle / p \rangle) \stackrel{\text{def}}{=} \delta_{\pi}(q, X), & q \in C_{\pi} \\
\delta_{\pi^*}(q, X) \stackrel{\text{def}}{=} \delta_{\pi}(q, X), & q \in Q_{\pi} \setminus C_{\pi} & \delta_{\pi^*}(q, \langle p \rangle) \stackrel{\text{def}}{=} \delta_{\pi}(q, X) \vee (0, q_{\pi^*,0}), & q \in C_{\pi}
\end{array}$$

for X in Σ and p in N. Because we assigned an odd priority to $q_{\pi^*,0}$, the automaton cannot loop indefinitely in $q_{\pi^*,0}$ and must eventually continue.

$\mathcal{A}_{\psi?}$ Define $Q_{\psi?} \stackrel{\text{def}}{=} Q_{\psi} \uplus \{q_{\psi?,0}, q_f\}$, $C_{\psi?} \stackrel{\text{def}}{=} \{q_f\}$, $c_{\psi?}$ extends c_{ψ} with $c_{\psi?}(q_{\psi?,0}) = c_{\psi?}(q_f) = 1$, and

$$\delta_{\psi?}(q_{\psi?,0}, X) \stackrel{\text{def}}{=} (0, q_f) \wedge (0, q_{\psi,0}), \quad \delta_{\psi?}(q_f, X) \stackrel{\text{def}}{=} \bot, \quad \delta_{\psi?}(q, X) \stackrel{\text{def}}{=} \delta_{\psi}(q, X),$$

for X in Σ and q in Q_{ψ}.

Finally, we consider the induction step for node formulæ:

$\mathcal{A}_{\langle\pi\rangle\psi}$ We construct the automaton by joining the critical states of $\mathcal{A}_\pi$ with the initial state of $\mathcal{A}_\psi$. Define $Q_{\langle\pi\rangle\psi} \stackrel{\text{def}}{=} Q_\pi \uplus Q_\psi$, $q_{\langle\pi\rangle\psi,0} \stackrel{\text{def}}{=} q_{\pi,0}$, $c_{\langle\pi\rangle\psi} \stackrel{\text{def}}{=} c_\pi \cup c_\psi$, and $\delta_{\langle\pi\rangle\psi}$ by

$$\delta_{\langle\pi\rangle\psi}(p,X) \stackrel{\text{def}}{=} \delta_\pi(p,X), \qquad p \in Q_\pi \setminus C_\pi \quad \delta_{\langle\pi\rangle\psi}(q,X) \stackrel{\text{def}}{=} \delta_\psi(q,X), \quad q \in Q_\psi$$
$$\delta_{\langle\pi\rangle\psi}(p,X) \stackrel{\text{def}}{=} \delta_\pi(p,X) \vee (0,q_{\psi,0}), \quad p \in C_\pi$$

for X in Σ.

$\mathcal{A}_{\psi_1\wedge\psi_2}$ We do a simple conjunction of the automata $\mathcal{A}_{\psi_1}$ and $\mathcal{A}_{\psi_2}$: define $Q_{\psi_1\wedge\psi_2} \stackrel{\text{def}}{=} Q_{\psi_1} \uplus Q_{\psi_2} \uplus \{q_{\psi_1\wedge\psi_2,0}\}$, $c_{\psi_1\wedge\psi_2} \stackrel{\text{def}}{=} c_{\psi_1} \cup c_{\psi_2} \cup \{(q_{\psi_1\wedge\psi_2,0},1)\}$, $\delta_{\psi_1\wedge\psi_2} \stackrel{\text{def}}{=} \delta_{\psi_1} \cup \delta_{\psi_2} \cup \{(q_{\psi_1\wedge\psi_2,0},X,(0,q_{\psi_1,0}) \wedge (0,q_{\psi_2,0})) \mid X \in \Sigma\}$.

$\mathcal{A}_{\neg\psi}$ We essentially construct the dual $\mathsf{dual}(\mathcal{A}_\psi)$ of $\mathcal{A}_\psi$: the latter accepts the complement of the language accepted by $\mathcal{A}_\psi$. However, we need to ensure that only opening nodes $\langle p\rangle$ are accepted, thus intersect with the automaton $\mathcal{A}_\top$ that only accepts opening nodes. Formally, $\mathsf{dual}(\mathcal{A}_\psi) = \langle Q_\psi, \Sigma, \delta_\psi, q_{\psi,0}, c_{\neg\psi}\rangle$ where $\delta_{\neg\psi}(q,X) \stackrel{\text{def}}{=} \mathsf{dual}(\delta_\psi(q,X))$ and $c_{\neg\psi}(q) \stackrel{\text{def}}{=} c_\psi(q)+1$ for all $q \in Q$ and $X \in \Sigma$. Here, dual is a function from $\mathbb{B}^+(Q \times \{-1,0,1\})$ to itself that applies the usual DeMorgan's law. It is easy to check that $\mathsf{dual}(\mathcal{A}_\psi)$ accepts the complement of $L(\mathcal{A}_\psi)$.

D ε-Free Case

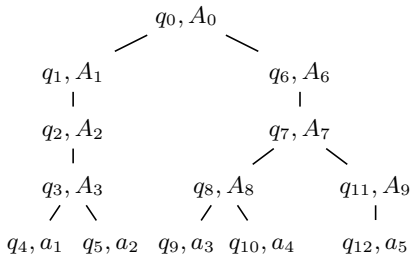
We prove in this section propositions 6 and 7, thus showing that the PFMC problem is PSPACE-complete in this case.

D.1 Upper Bound: Proposition 6

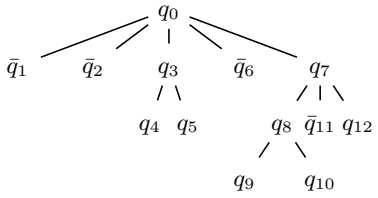
Proof. Let $G = \langle N, \Sigma, P, S\rangle$, w be a string in Σ^*, and φ be a $\text{PDL}_{\mathsf{tree}}$ formula. Without loss of generality, we assume G to have productions with right-parts of length at most 2; since G is ε-free, these right-parts have length at least one. We want to construct a non-recursive DTD $D = \langle N', P', S'\rangle$ and a $\text{PDL}_{\mathsf{tree}}$ formula φ' s.t. the parse forest model checking problem on G, w, and φ has a solution iff φ is satisfiable in presence of D, thereby reducing our instance to an instance of a problem in PSPACE by Proposition 4.

We want to construct D from the polynomial-sized automaton $\mathcal{A}_{G,w}$ by removing chains of *unit* rules $q \to A(q')$ of $\mathcal{A}_{G,w}$; recall that this automaton uses states of form $q = (i,X,j)$ where $0 \le i < j \le |w|$ and X is in V. Let for this $\bar{Q}_{G,w}$ be a disjoint copy of $Q_{G,w}$ and define $N' \stackrel{\text{def}}{=} \bar{Q}_{G,w} \uplus Q_{G,w}$.

Chain Sequences. For each q in $Q_{G,w}$, we consider the set of sequences of successive states $q = q_0, q_1, \ldots, q_n$ we can visit using only unit rules $q_i \to A_i(q_{i+1})$ of $\delta_{G,w}$ and such that q_n has a binary rule $q_n \to A_n(q'\,q'')$ or a nullary rule $q_n \to a()$ in $\delta_{G,w}$. More precisely, we are interested in the relabeled sequence $\bar{q} = \bar{q}_0, \bar{q}_1, \ldots, \bar{q}_{n-1}, q_n$ of copies $\bar{q}_i$ of q_i, except on the very last position. We call $\mathsf{chains}(q)$ the language of such sequences. Formally, $\mathsf{chains}(q)$ is a regular language over N' that we can define thanks to a NFA $\mathcal{A}_q \stackrel{\text{def}}{=} \langle N', N', \delta_q, \{\bar{q}\}, Q_{G,w}\rangle$



(a) A run of $\mathcal{A}_{Q,w}$ on a tree in the parse forest of w according to G.



(b) The corresponding tree in $L(D)$.

Figure 5: The tree transformation for the proof of Proposition 6.

with state space N' where

$$\begin{aligned}\delta_q \stackrel{\text{def}}{=} &\{(\bar{p}, \bar{p}, \bar{p}') \mid \exists A \in N, p \to A(p') \in \delta_{G,w}\} \\ &\cup \{(\bar{p}, p, p) \mid \exists A \in N, \exists p_1, p_2 \in Q_{G,w}, p \to A(p_1\, p_2) \in \delta_{G,w}\} \\ &\cup \{(\bar{p}, p, p) \mid \exists a \in \Sigma, p \to a() \in \delta_{G,w}\}\ .\end{aligned}$$

Note that $\mathcal{A}_q$ has a size linear in that of $\mathcal{A}_{G,w}$. We can see chains as a regular substitution from $Q^*_{G,w}$ to N'^* by setting $\mathsf{chains}(\varepsilon) \stackrel{\text{def}}{=} \varepsilon$ and $\mathsf{chains}(uv) \stackrel{\text{def}}{=} \mathsf{chains}(u)\mathsf{chains}(v)$ for all u, v in $Q^*_{G,w}$.

The DTD. We can now express the productions P' of D:

$$P(\bar{q}) \stackrel{\text{def}}{=} \emptyset \qquad P(q) \stackrel{\text{def}}{=} \bigcup_{q \to A(q_1\, q_2) \in \delta_{G,w}} \mathsf{chains}(q_1\, q_2) \cup \bigcup_{q \to a() \in \delta_{G,w}} \varepsilon\ .$$

Thus, the symbols in $\bar{Q}_{G,w}$ are non-productive and only employed to represent a chain sequence that has been transformed into a sequence of siblings in the DTD. Also, because any word in $\mathsf{chains}(q)$ for some q is of form up with u in $\bar{Q}^*_{G,w}$ and p in $Q_{G,w}$, any internal node in a tree of D has exactly two children labeled by states in $Q_{G,w}$. Therefore, and because G is ε-free, we get that D is non-recursive. See Figure 5 for an illustration of the tree transformation we operated.

The Formula. It remains to define a formula φ' that will be interpreted on the transformed trees of D. For this, we need to interpret the atomic propositions in $\mathsf{AP} = V$ over the new set of labels N', and to interpret the child $\downarrow$ and sibling $\to$ relations.

Regarding the atomic propositions, we can interpret a label X in V as

$$\bigvee_{0 \le i < j \le |w|} (i, X, j) \vee \overline{(i, X, j)} \qquad \text{(interpretation of } X\text{)}$$

over N'.

Regarding the relations, we first define $\mathsf{bar} \stackrel{\text{def}}{=} \bigvee_{\bar{q} \in \bar{Q}_{G,w}}$ to help us differentiate between "rotated" nodes and preserved ones. We then interpret $\downarrow$ as a

disjunction of paths depending on whether we are on a rotated node, where the test $[\leftarrow]\neg\mathsf{bar}$ allows to check that the current node is an "original" child:

$$(\neg\mathsf{bar}?;\downarrow;[\leftarrow]\neg\mathsf{bar}) + (\mathsf{bar}?;\rightarrow) \qquad \text{(interpretation of } \downarrow)$$

For $\rightarrow$, we set:

$$([\leftarrow]\neg\mathsf{bar}?);(\mathsf{bar}?;\rightarrow)^*;\neg\mathsf{bar}?;\rightarrow \ . \qquad \text{(interpretation of } \rightarrow)$$

The initial test prevents the nodes taken from a chain from having a right sibling; then the test sequence advances to the end of the chain before we make the actual move to the original right sibling.

We can conclude by noting that both D and φ' can be computed in time polynomial in the size of the input and invoking Proposition 4. □

D.2 Lower Bound: Proposition 7

Proof. We reduce from the membership problem of linear bounded automata (LBA). Suppose we are given an LBA $M = \langle Q, \Gamma, \Sigma, \delta, q_1, F\rangle$ with state set Q, tape alphabet Γ, input alphabet $\Sigma \subseteq \Gamma$, transition relation $\delta \subseteq Q \times \Gamma \times Q \times \Gamma \times \{-1, 0, 1\}$, initial state $q_1 \in Q$, and set of final states $F \subseteq Q$. Let $Q = \{q_1, \ldots, q_\ell\}$; we assume that $\Gamma = \{a_1, a_2, \ldots, a_m\}$ contains two endmarkers $a_1 = \triangleleft$ and $a_2 = \triangleright$ that surround the input and are never erased nor crossed during the run of the machine.

We are also given a string $x = b_1 b_2 \cdots b_n$ with each $b_i \in \Sigma$; $b_1 = \triangleleft$ and $b_n = \triangleright$. We have to decide whether x is accepted by M. We are going to construct a word w, a CFG G, and a $\mathrm{PDL}_{\mathsf{core}}[\downarrow]$ formula φ, s.t. the PFMC problem has a solution for $\langle w, G, \varphi\rangle$ iff M accepts x.

Encoding as Linear Trees. A configuration of M is a sequence of length n of form $\triangleleft\gamma q\gamma'\triangleright$ where q is the current state in Q, $\triangleleft\gamma\gamma'\triangleright$ is the current tape contents, and $|\triangleleft\gamma| = h$ indicates that the head is currently on the last symbol of $\triangleleft\gamma$, i.e. the hth symbol of the tape.

We encode such a configuration by a contiguous sequence α of nodes as follows:

- The first node is S and it is followed by a sequence of n nodes, among which one is labeled H and the others $\bar{H}$; the position of H in this sequence denotes the position of the head in the configuration of M.
- This sequence is followed by a sequence of ℓ nodes, one labeled C and the others $\bar{C}$, which together describe the current state as q_k if the occurrence of C is the kth symbol in the sequence.
- Then we encode the tape contents as n successive sequences each of length m of nodes, with each time one labeled A and the others $\bar{A}$. The ith such sequence encodes the contents of the ith cell of the tape of M with A occurring at the jth position indicating that this cell contains a_j.

Thus α is of length $1 + n + \ell + nm$.

String and Grammar. We *fix* $w \stackrel{\text{def}}{=} a$ and $G \stackrel{\text{def}}{=} \langle \{S, H, \bar{H}, A, \bar{A}, C, \bar{C}\}, \{a\}, P, S\rangle$ with productions

$$\begin{array}{lll} S \to H \mid \bar{H} & & \\ H \to H \mid \bar{H} \mid C \mid \bar{C} & C \to C \mid \bar{C} \mid A \mid \bar{A} & A \to A \mid \bar{A} \mid S \mid a \\ \bar{H} \to H \mid \bar{H} \mid C \mid \bar{C} & \bar{C} \to C \mid \bar{C} \mid A \mid \bar{A} & \bar{A} \to A \mid \bar{A} \mid S \mid a\ . \end{array}$$

Therefore, the trees in the parse forest $L_{G,w}$ are essentially sequences over $N^* \cdot \{a\}$. Clearly, all the encodings of finite runs of any LBA M will be in this set; it will be the formula's task to look for an accepting run of our particular M on x among all these trees.

The Formula $\varphi_{M,x}$. Let us turn to the definition of our PDL$_{\mathsf{core}}[\downarrow]$ formula. We start by defining low-level formulæ useful for testing the properties of the current configuration: assume we are on an S-labeled node:

$$\mathsf{h}(h) \stackrel{\text{def}}{=} (\langle\downarrow^h\rangle H) \wedge \bigwedge_{i \in \{1,\ldots,n\}\setminus\{h\}} \langle\downarrow^j\rangle \bar{H}$$

tests whether the head is at position h. In the same way,

$$\mathsf{q}(k) \stackrel{\text{def}}{=} \langle\downarrow^n\rangle\big((\langle\downarrow^k\rangle C) \wedge \bigwedge_{k' \in \{1,\ldots,\ell\}\setminus\{k\}} \langle\downarrow^{k'}\rangle \bar{C}\big)$$

then tests whether the current state is q_k, and

$$\mathsf{p}(i,j) \stackrel{\text{def}}{=} \langle\downarrow^{n+\ell+im}\rangle\big((\langle\downarrow^j\rangle A) \wedge \bigwedge_{j' \in \{1,\ldots,m\}\setminus\{j\}} \langle\downarrow^{j'}\rangle \bar{A}\big)$$

tests whether the i position on the tape is symbol a_j. Finally, we can go to the next configuration by the path

$$\mathsf{next} \stackrel{\text{def}}{=} \downarrow^{n+\ell+nm}\ .$$

We can now check that a parse tree of $L_{G,w}$ is really the encoding of an accepting run of M on x. First, at each S node, we should find a full configuration:

$$\varphi_{conf} \stackrel{\text{def}}{=} [\downarrow^*]S \supset \bigvee_{h=1}^{n} \mathsf{h}(h) \wedge \bigvee_{k=1}^{\ell} \mathsf{q}(k) \wedge \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} \mathsf{p}(i,j) \wedge \langle\mathsf{next}\rangle a \vee S\ .$$

The initial configuration should have its head on the initial position 1, be in the initial state q_1, and have $x = b_1 \cdots b_n$ as tape contents:

$$\varphi_{init} \stackrel{\text{def}}{=} S \wedge \mathsf{h}(1) \wedge \mathsf{q}(1) \wedge \bigwedge_{i=1}^{n} \mathsf{p}(i, b_i)\ .$$

The leaf of the tree should be reached in a final configuration:

$$\varphi_{final} \stackrel{\text{def}}{=} [\downarrow^*](S \wedge \langle\mathsf{next}\rangle a) \supset \bigvee_{q_k \in F} \mathsf{q}(k)\ .$$

Successive configurations should respect the transition relation:

$$\varphi_{trans} \stackrel{\text{def}}{=} [\downarrow^*](S \wedge \neg\langle \mathsf{next}\rangle a) \supset \bigvee_{h=1}^{n} \bigvee_{k=1}^{\ell} \bigvee_{c=1}^{m} \bigvee_{(q_k, a_c, q_{k'}, a_{c'}, d) \in \delta} \big(\mathsf{h}(h) \wedge \mathsf{q}(k) \wedge \mathsf{p}(h,c)$$
$$\wedge \big(\bigwedge_{h \neq i=1}^{n} \bigvee_{j=1}^{m} \mathsf{p}(i,j) \wedge \langle \mathsf{next}\rangle \mathsf{p}(i,j)\big) \wedge \langle \mathsf{next}\rangle(\mathsf{h}(h+d) \wedge \mathsf{q}(k') \wedge \mathsf{p}(h,c'))\big) .$$

We finally define our PDL$_{\mathsf{core}}[\downarrow]$ formula as the conjunction of the previous formulæ:

$$\varphi_{M,x} \stackrel{\text{def}}{=} \varphi_{conf} \wedge \varphi_{init} \wedge \varphi_{final} \wedge \varphi_{trans} \ .$$

To conclude, we observe that a tree in $L_{G,w}$ is a model of $\varphi_{M,x}$ iff there is an accepting run of M on x. As G and w are fixed and $\varphi_{M,x}$ can be computed in space logarithmic in the size of $\langle M, x\rangle$, this proves the PSPACE-hardness of the PFMC problem in the ε-free case. □